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1992 J. Phys. A: Math. Gen. 25 5761

(http://iopscience.iop.org/0305-4470/25/21/029)

J. Phys. A: Math. Gen. **25** (1992) 5761-5777. Printed in the UK

Relative convergences of the WKB and SWKB approximations

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Received 5 August 1991, in final form 29 May 1992

Abstract. The relative convergences of the WKB and SWKB approximations are examined by calculating the eigenenergies for four potentials by one- and two-term WKB and SWKB approximations. Exact eigenenergies for these four potentials are also calculated by a numerical integration of the Schrödinger equation. Varied results are found for the four potentials. It is found that in general the effect of the second term in the WKB and SWKB approximations depends on the potential, the parameters involved and the quantum number of the state. No simple generalizations are possible.

1. Introduction

The application of supersymmetric quantum mechanics (SUSYQM) (Witten 1981, Cooper and Freedman 1983) to bound state problems has led to a number of interesting results. Comtet, Bandrauk and Campbell (1985) showed that the structure of SUSYQM motivates a modification of the conventional WKB quantization condition. They further found that this modified condition, now called the supersymmetric WKB (SWKB) quantization rule, gives the exact energy eigenvalues in the first order for several solvable potentials. Khare (1985) found similar results for three other solvable potentials. Dutt *et al* (1986) showed that the leading order SWKB condition will always reproduce the exact bound-state spectrum for any shape-invariant potential (Gedenshtein 1983). Raghunathan *et al* (1987) showed that for the Rosen–Morse potential, which is a solvable potential, all higher-order corrections in the SWKB scheme vanish. Dutt *et al* (1991) have reviewed the lowest order SWKB approximation. The question of the effect of higher-order SWKB approximation for a potential which is not exactly solvable was considered by Dutt *et al* (1987), who compared the results for a potential due to Murrell (1969) by one- and two-term SWKB with one- and two-term WKB. They found that one-term SWKB values are much closer to the exact values than the one-term WKB values. The trend continued even for the two-term values indicating that perhaps the SWKB expansion (in orders of $\hbar^2$) has better convergence than the corresponding one in the old WKB approach. Higher-order terms in the conventional WKB method had been obtained by Kesarwani and Varshni (1978, 1980, 1981, 1982a) and this was done for the SWKB method by Adhikari *et al* (1988). These authors obtained energy eigenvalues by SWKB method up to order $\hbar^6$ for the following two potentials

$$V(x)=x^2+\tfrac{1}{9}x^6 \tag{1}$$

$$V(x)=x^{10/3}+\tfrac{5}{3}x^{2/3}. \tag{2}$$

Results for the potential (1) were also obtained by Vasan *et al* (1988).

A number of authors (Dutt *et al* 1987, Roy *et al* 1988, Fricke *et al* 1988, Khare and Varshni 1989, DeLaney and Nieto 1990) have compared one-term SWKB results with one-term WKB results for a variety of potentials. The present situation may be summarized as follows. For shape-invariant solvable potentials, SWKB gives exact results in all cases, while WKB gives exact results only for the harmonic oscillator and the Morse potential. For all other types of potentials, broadly speaking, for $n = 0$, the SWKB results are better than the WKB ones in most cases, for $n = 1, 2, 3$ the results are mixed, for $n > 3$ for some potentials SWKB is better, while for others WKB is better. One, of course, has to bear in mind that given any potential, the WKB answer can be immediately computed, while the SWKB answer can only be obtained if we also know the corresponding superpotential $W(x)$ which may not always be known.

Clearly the vexed question of the role played by higher-order terms comes to mind. There is no easy way to examine the relative convergences of the WKB and SWKB approximations. Detailed investigations are required with individual potentials. The use of the term 'convergence' in this context needs some qualification and clarification. It is known that, in general, the WKB series does not converge but is, instead, an asymptotic expansion (Birkhoff 1933, Kemble 1958, Bohm 1951, Bender and Orszag 1978). This means that the magnitude of the terms may diminish up to a certain term but after that it may begin to increase. This would mean that for any given value of n, successive approximations to the nth eigenvalue obtained by taking more and more terms should improve to some maximal accuracy and then become worse. Examples of such behaviour have been previously recorded (Bender and Orszag 1978, Kesarwani and Varshni 1981, 1982a, b, c). Thus the concept of convergence is meaningful only up to and including the term inclusion of which leads to an improvement in the eigenvalue. Here we consider this question in a limited form. What will happen if we take the second term in both SWKB and WKB into account? The results will indicate which of the two series has the better convergence. The only investigation that throws some light on this question is that of Dutt *et al* (1987) who compared two-term SWKB results with two-term WKB results for the Murrell potential. It turned out that both sets of results were almost the same and every close to the 'exact' results. In the present paper we obtain eigenvalues by the two-term SWKB and WKB approximation for four different potentials in order to obtain better evidence to answer the aforesaid question. In this paper we shall only consider single-well potentials. Some of the potentials examined in this paper become double- or triple-well potentials for a range of values of the parameters concerned; we shall exclude such values of the parameters. We shall use units such that $\hbar = 2m = 1$.

2. The four potentials

2.1. First potential

The non-polynomial oscillator represented by the potential

$$V(x) = x^2 + \lambda x^2/(1 + gx^2) \tag{3}$$

where λ and g are parameters, has been investigated by a variety of techniques in recent years (Varshni 1987 and references therein, Adhikari *et al* 1991 and references given therein, Bose and Varma 1989, Filho and Ricotta 1989, Fu-Bin 1989, Estrin *et al* 1990, Ifantis and Panagopoulos 1990, Pons and Marcilhary 1991). Roy *et al* (1988) showed that potential (3) becomes supersymmetric provided λ and g are constrained by a certain relation. Roy *et al* (1988) suggested the following superpotential

$$W(x)=\mu x-2gx/(1+gx^2). \tag{4}$$

In SUSYQM

$$V_{\pm}(x)=W^2(x)\pm W'(x). \tag{5}$$

Thus

$$V_{-}(x)=\mu^2x^2+\frac{2g+4\mu}{1+gx^2}-5\mu. \tag{6}$$

If (6) and (3) are to be the same, then we must have

$$-\lambda/g=(2g+4\mu) \tag{7}$$

$$\mu=\pm 1. \tag{8}$$

These are the constraints that λ and g must satisfy.

The energy of the non-polynomial oscillator (E) is connected to E_- obtained from $W(x)$ by

$$E=E_-+5\mu+\lambda/g. \tag{9}$$

When $\mu=+1$, an exact analytic expression for the ground state can be obtained:

$$E^0=\lambda/g+5. \tag{10}$$

But for $\mu=-1$, $E_-\neq 0$, and an exact expression is not possible.

2.2. *Supersymmetric potentials*

We shall generate the other three potentials by supersymmetric quantum mechanics. There are two ways of doing it. Either one can start with a suitable form of the ground state wavefunction or one can start with the superpotential (Boya *et al* 1987, Dutt *et al* 1988, Roy *et al* 1991). In the latter case one has to ensure that the corresponding wavefunction is normalizable. We shall use the former procedure.

2.3. *Second potential*

We assume the ground-state wavefunction to be given by

$$\phi_0(x)=\exp(-ax^2-bx^4) \tag{11}$$

where a and b are constants. Then the superpotential $W(x)$ is obtained from

$$\begin{aligned}W(x)&=-\phi_0'(x)/\phi_0(x)\\&=2ax+4bx^3.\end{aligned} \tag{12}$$

Thus the potential is

$$\begin{aligned}V_{-}(x)&=W^2(x)-W'(x)\\&=(4a^2-12b)x^2+16abx^4+16b^2x^6-2a.\end{aligned} \tag{13}$$

The ground-state energy for this potential is of course zero. If $a^2 > 3b$, potential (13) has one minimum, and when $a^2 < 3b$ it has three minima. We are interested in the one-minimum case only and so we shall choose a and b such that $a^2 > 3b$. Potentials of the type $V(x) = \alpha x^2 + \beta x^4 + \gamma x^6$ have been investigated by a variety of methods. The following are a few of the recent references: Znojil (1986), Dutta and Willey (1988), Burrows *et al* (1989), Adhikari *et al* (1989), Chaudhuri and Mondal (1989, 1991), Singh *et al* (1990).

2.4. *Third potential*

$$\phi_0(x) = \frac{\exp(-ax^2)}{(1+bx^2)} \tag{14}$$

$$W(x) = \frac{2x(a+b+abx^2)}{1+bx^2} \tag{15}$$

$$V_-(x) = \frac{2[-a-b+(2a^2+3b^2+2ab)x^2+(3ab^2+4a^2b)x^4+2a^2b^2x^6]}{(1+bx^2)^2}. \tag{16}$$

If $b/a > 7.565\,96$, this potential also becomes a three-minimum potential. To avoid it, we have chosen a and b such that $b/a < 7.565\,96$.

2.5. *Fourth potential*

$$\phi_0(x) = \exp(-ax^2) + \exp(-bx^2) \tag{17}$$

$$W(x) = \frac{2x\{a+b\exp[(a-b)x^2]\}}{1+\exp[(a-b)x^2]} \tag{18}$$

$$V_-(x) = \frac{\{-2a+4a^2x^2+(-2b+4b^2x^2)\exp[(a-b)x^2]\}}{\{1+\exp[(a-b)x^2]\}}. \tag{19}$$

The expressions are symmetric with respect to a and b. We shall take $b > a$. If $b/a > 3.68$, this potential also develops three minima. Hence a and b were given values such that $b/a < 3.68$.

3. Two-term WKB and SWKB

The two-term WKB (Krieger *et al* 1967, Kesarwani and Varshni 1978) and SWKB (Dutt *et al* 1987, Adhikari *et al* 1988) expressions have been derived by previous workers, so here we shall merely quote them.

WKB:
$$\int_a^b (E-V)^{1/2}\,\mathrm{d}x - \frac{1}{24}\frac{\mathrm{d}}{\mathrm{d}E}\int_a^b \frac{V''}{(E-V)^{1/2}}\,\mathrm{d}x = (n+\tfrac{1}{2})\pi \tag{20}$$

where a and b are the turning points defined by $E - V = 0$.

SWKB:
$$\int_c^d [E_- - W^2]\,\mathrm{d}x - \frac{1}{24}\int_c^d \frac{[2W'^2 - WW'']}{(E_- - W^2)^{3/2}} = n\pi \tag{21}$$

where c and d are the turning points of $E_- - W^2 = 0$. The one-term results are obtained by dropping the second term on the left-hand side of equations (20) and (21).

4. Results and discussion

The numerical method of Barwell *et al* (1979) was used to evaluate the second integral on the left-hand side of equations (20) and (21). An iterative method was used to calculate the energy. The 'exact' energy was also calculated by numerical integration of the Schrödinger equation. We shall use the following acronyms: 1WKB for one-term WKB, 2WKB for two-term WKB, and similarly for the SWKB.

4.1. *First potential, equation (3)*

Equations (7) and (8) put important constraints on the possible values of λ and g. First let us consider the case when $\mu=+1$. Then equation (7) becomes

$$-\lambda/g=2g+4. \tag{22}$$

For positive g, λ is negative. λ is acceptable between 0 and -1, but if $\lambda<-1$, then potential (3) becomes a double minimum potential. $\lambda=-1$ when $g=0.224\,745$. Thus the allowed values are $0.224\,745>g\geqslant 0$ and $0\geqslant\lambda>-1$. We shall call it region I. Next we consider the case when $\mu=-1$. Equation (7) becomes

$$-\lambda/g=2g-4. \tag{23}$$

By arguments similar to those given for the $\mu=+1$ case, it can be readily seen that these are three possible allowed regions:

II. $1\geqslant g\geqslant 0$, $2\geqslant\lambda\geqslant 0$;
III. $2\geqslant g>1$, $2>\lambda\geqslant 0$;
IV. $2.224\,745>g>2$, $0>\lambda>-1$.

Roy *et al* (1988) have considered regions I and II only. Here we shall present results for all the four allowed regions. In regions I and II, Roy *et al* (1982) have used certain sets of values of g and λ for comparing 1WKB and 1SWKB results for the potential (3) and the same sets of values of g and λ were used by us so that the results could be compared. The results are shown in table 1 for $\mu=+1$ (region I), and in table 2 for $\mu=-1$ (region II). When $\mu=+1$, E_- for the ground state is zero, and the 1SWKB gives the exact energy. In such a situation, the numerical evaluation of the second term on the left-hand side of equation (21) is subject to considerable uncertainties because there is a singularity right at $E_-=0$ and the region of integration is very small. Hence in the 2SWKB column in table 1, for $n=0$, the energy shown is from 1SWKB and it is enclosed in parentheses. When $\mu=-1$, in equation (21), on the right-hand side, n is replaced by $(n+1)$.

The results for regions III and IV are shown in tables 3 and 4 respectively. The tabular arrangement of the results is slightly different from that of tables 1 and 2, because in tables 3 and 4 we also include the 1WKB and 1SWKB results. The percentage errors with respect to the exact energy are shown immediately below the WKB and SWKB results. The best value amongst the four WKB and SWKB results is marked by an asterisk. We shall discuss tables 1 and 2 together.

Tables 1 and 2. A comparison of the two-term results obtained here with the one-term results obtained by Roy *et al* (1988) shows that in practically all cases the inclusion of the second term has led to an improvement in the energy, both for WKB and SWKB. Sometimes, the improvement is indeed remarkable, for example, for $g=1$, $\lambda=2$, $n=0$, the error in the one-term SWKB energy was 8.236%, and with two-terms, it is only

Table 1. 2WKB, 2SWKB and exact eigenenergies for the potential (3) for $\mu=+1$ and $E_-=0$ for the ground state. Region I of the parameters.

g	λ	n	2WKB	Percentage error	2SWKB	Percentage error	Exact
0.05	−0.205	0	0.899 92	−0.009	(0.900 00)	0.000	0.900 00
		1	2.714 50	−0.001	2.714 54	0.001	2.714 52
		2	4.554 60	0.000	4.554 62	0.000	4.554 60
		3	6.414 41	0.000	6.414 41	0.000	6.414 40
		4	8.289 94	0.000	8.289 93	0.000	8.289 92
		5	10.178 31	0.000	10.178 29	0.000	10.178 28
0.10	−0.420	0	0.798 24	−0.220	(0.800 00)	0.000	0.800 00
		1	2.455 89	0.008	2.456 12[a]	0.017	2.455 70
		2	4.198 15	0.006	4.198 05	0.004	4.197 90
		3	5.991 85	0.007	5.991 54	0.002	5.991 40
		4	7.820 79	0.009	7.820 28	0.002	7.820 10
		5	9.675 50	0.010	9.674 79	0.003	9.674 54
0.15	−0.645	0	0.689 87	−1.447	(0.700 00)	0.000	0.700 00
		1	2.222 34	0.128	2.221 96	0.111	2.219 50
		2	3.906 53	0.051	3.904 87	0.009	3.904 52
		3	5.667 94	0.056	5.665 44	0.012	5.664 76
		4	7.477 78	0.055	7.474 52	0.012	7.473 65
		5	9.320 98	0.054	9.317 01	0.011	9.315 98
0.20	−0.880	0	0.571 18	−4.804	(0.600 00)	0.000	0.600 00
		1	2.014 95	0.643	2.009 70	0.381	2.002 08
		2	3.664 49	0.213	3.657 09	0.011	3.656 70
		3	5.409 02	0.203	5.400 12	0.039	5.398 04
		4	7.210 23	0.173	7.200 01	0.031	7.197 80
		5	9.049 17	0.152	9.037 79	0.026	9.035 46

[a] Indicates that the 2SWKB result is poorer than the 2WKB result.

0.067%. The cases where the 2SWKB energy is worse than the 2WKB energy are shown by a superscript a in the 2SWKB energy column. In the one-term results there were only 15 cases (out of 60) for which the SWKB results were better than the WKB ones; with two terms this number has shot up to 52. In two additional cases, the 2SWKB is only marginally worse than 2WKB. It clearly shows that the second-term plays a vital role and the convergence of the SWKB series appears to be substantially better than that of the WKB series, at least for the regions I and II of the potential (3).

It will be noticed from tables 1 and 2 that there is a tendency for the errors to increase with the increase in the numerical magnitudes of g and λ. It is of some interest to note that of the eight cases for which the 2SWKB results are poorer than the 2WKB ones, five are for $n=1$.

Table 3. For the WKB results it will be noticed that the inclusion of the second term has led to an improvement in the energy only for $n=0$ state for the four sets of parameters, otherwise, in all other cases it worsens the energy indicating that the WKB series becomes divergent for $n>0$. For the SWKB cases we notice that for $n=0$ there is a large error in the energy which decreases very sharply with the inclusion of the second term, inasmuch as the 2SWKB results are the best ones for $n=0$. For the first set of parameters, the inclusion of the second term has improved the energy for $n=1$ and 2 also, but for the other three sets of parameters the 2SWKB results are worse than

Table 2. 2WKB, 2SWKB and exact eigenenergies for the potential (3) for $\mu = -1$ and $E_- = 0$ for the ground state. Region II of the parameters.

g	λ	n	2WKB	Percentage error	2SWKB	Percentage error	Exact
0.10	0.380	0	1.157 19	0.016	1.156 97	−0.004	1.157 01
		1	3.440 22	0.001	3.440 11[a]	−0.002	3.440 17
		2	5.669 84	0.000	5.669 79[a]	−0.001	5.669 85
		3	7.859 97	−0.001	7.859 96	−0.001	7.860 03
		4	10.020 23	−0.001	10.020 27	−0.001	10.020 37
		5	12.157 41	−0.002	12.157 51	−0.001	12.157 65
0.20	0.720	0	1.262 69	0.084	1.261 26	−0.030	1.261 63
		1	3.703 34	0.000	3.702 95[a]	−0.011	3.703 35
		2	6.014 16	−0.008	6.014 17	−0.008	6.014 66
		3	8.244 89	−0.015	8.245 38	−0.009	8.246 10
		4	10.423 63	−0.021	10.424 87	−0.009	10.425 82
		5	12.566 90	−0.026	12.569 14	−0.008	12.570 19
0.30	1.020	0	1.338 49	0.198	1.334 88	−0.072	1.335 84
		1	3.873 24	−0.017	3.872 64[a]	−0.033	3.873 90
		2	6.208 86	−0.038	6.209 60	−0.027	6.211 25
		3	8.438 65	−0.062	8.441 62	−0.026	8.443 85
		4	10.607 31	−0.079	10.613 49	−0.020	10.615 65
		5	12.737 77	−0.088	12.747 62	−0.010	12.748 95
0.40	1.280	0	1.393 73	0.351	1.387 25	−0.116	1.388 86
		1	3.981 86	−0.059	3.981 35[a]	−0.072	3.984 23
		2	6.313 79	−0.104	6.316 69	−0.058	6.320 36
		3	8.526 99	−0.152	8.536 04	−0.046	8.539 95
		4	10.678 22	−0.175	10.695 16	−0.017	10.696 94
		5	12.793 25	−0.180	12.818 17	0.015	12.816 29
0.50	1.500	0	1.433 37	0.538	1.423 54	−0.152	1.425 70
		1	4.045 64	−0.137	4.045 69	−0.136	4.051 20
		2	6.357 97	−0.212	6.365 14	−0.100	6.371 50
		3	8.548 87	−0.284	8.568 75	−0.052	8.573 19
		4	10.681 98	−0.300	10.716 54	0.022	10.714 15
		5	12.782 99	−0.289	12.830 93	0.085	12.820 01
1.00	2.000	0	1.475 74	1.964	1.448 29	0.067	1.447 32
		1	3.946 97	−1.286	3.954 28	−1.103	3.998 40
		2	6.100 17	−1.268	6.167 67	−0.175	6.178 49
		3	8.195 77	−1.195	8.354 30	0.716	8.294 90
		4	10.270 30	−0.947	10.498 05[a]	1.250	10.368 48
		5	12.331 22	−0.753	12.600 20[a]	1.412	12.424 76

[a] Indicates that the 2SWKB result is poorer than the 2WKB result.

the 1SWKB ones for $n > 0$. It is of some interest to see that for $n > 0$, the 1WKB results are the best.

Table 4. Broadly speaking the pattern of the results is similar to that in table 3, but there are some important differences. Here all 2WKB results are worse than the 1WKB results. Like the previous case the 1SWKB energy for $n = 0$ has a large error for all the four sets of parameters, and the error is sharply reduced with the inclusion of the second term, making the 2SWKB results the best ones. However, for $n > 0$ the 2SWKB results are worse than the 1SWKB ones in all other cases. Here also we find that for $n > 0$, the 1WKB results are the best ones.

Table 3. WKB, SWKB and exact eigenenergies for the potential (3) for $\mu=-1$. The first line against each quantum number gives the energies and the second one, the corresponding percentage errors. The best values are marked by an asterisk. Region III of the parameters.

g	λ	n	1WKB	2WKB	1SWKB	2SWKB	Exact
1.20	1.92	0	1.482 50	1.438 06	1.248 73	1.407 47*	1.401 35
			5.791	2.620	−10.891	0.437	
		1	3.844 31*	3.785 63	3.783 87	3.788 05	3.865 10
			−0.538	−2.056	−2.102	−1.994	
		2	5.992 91*	5.888 51	5.964 50	5.997 11	5.991 56
			0.023	−1.720	−0.452	0.093	
		3	8.078 01*	7.964 03	8.060 82	8.219 27	8.081 83
			−0.047	−1.458	−0.260	1.701	
		4	10.134 88*	10.026 84	10.123 07	10.376 09	10.134 98
			−0.001	−1.067	−0.118	2.379	
		5	12.176 35*	12.078 66	12.167 59	12.475 01	12.177 58
			−0.010	−0.812	−0.082	2.442	
1.40	1.68	0	1.402 39	1.370 91	1.144 34	1.344 34*	1.331 35
			5.336	2.972	−14.047	0.976	
		1	3.669 34*	3.587 64	3.603 87	3.567 09	3.691 04
			−0.588	−2.801	−2.362	−3.358	
		2	5.775 03*	5.656 68	5.743 15	5.815 74	5.771 98
			0.053	−1.998	−0.499	0.758	
		3	7.835 15*	7.718 35	7.815 38	8.094 21	7.839 00
			−0.049	−1.539	−0.301	3.256	
		4	9.875 21*	9.770 71	9.861 40	10.262 06	9.874 73
			0.005	−1.053	−0.135	3.922	
		5	11.904 37*	11.813 25	11.894 02	12.352 60	11.905 57
			−0.010	−0.775	−0.097	3.755	
1.60	1.28	0	1.293 48	1.271 15	1.017 56	1.259 28*	1.239 95
			4.317	2.516	−17.936	1.559	
		1	3.466 29*	3.373 85	3.395 28	3.296 42	3.484 91
			−0.534	−3.187	−2.572	−5.409	
		2	5.533 25*	5.422 45	5.497 41	5.644 04	5.529 52
			0.067	−1.936	−0.581	2.071	
		3	7.571 22*	7.470 93	7.548 54	7.990 41	7.574 59
			−0.045	−1.369	−0.344	5.490	
		4	9.596 49*	9.510 88	9.580 44	10.160 74	9.595 69
			0.008	−0.884	−0.159	5.889	
		5	11.614 86*	11.542 30	11.602 72	12.236 11	11.615 92
			−0.009	−0.634	−0.114	5.339	
1.80	0.72	0	1.158 42	1.139 52	0.870 18	1.149 60*	1.129 04
			2.602	0.928	−22.927	1.821	
		1	3.241 61*	3.168 33	3.164 28	2.976 31	3.253 00
			−0.350	−2.603	−2.727	−8.506	
		2	5.273 54*	5.199 92	5.233 21	5.502 63	5.270 69
			0.054	−1.343	−0.711	4.401	
		3	7.291 63*	7.229 72	7.265 68	7.912 91	7.293 78
			−0.029	−0.878	−0.385	8.488	
		4	9.303 65*	9.252 79	9.285 12	10.072 70	9.302 96
			0.007	−0.539	−0.192	8.274	
		5	11.312 39*	11.270 23	11.298 28	12.126 07	11.313 08
			−0.006	−0.379	−0.131	7.186	

Table 4. WKB, SWKB and exact eigenenergies for the potential (3) for $\mu=-1$. The first line against each quantum number gives the energies and the second one, the corresponding percentage errors. The best values are marked by an asterisk. Region IV of the parameters.

g	λ	n	1WKB	2WKB	1SWKB	2SWKB	Exact
2.05	−0.205	0	0.957 11	0.973 81	0.658 87	0.970 11*	0.965 04
			−0.821	0.909	−31.726	0.526	
		1	2.937 39*	2.967 25	2.850 83	2.498 00	2.933 89
			0.119	1.137	−2.831	−14.857	
		2	4.929 82*	4.954 49	4.883 12	5.400 33	4.930 87
			−0.021	0.479	−0.969	9.521	
		3	6.925 53*	6.944 80	6.895 02	7.852 15	6.924 83
			0.010	0.288	−0.431	13.391	
		4	8.922 68*	8.937 94	8.900 70	9.978 74	8.922 96
			−0.003	0.168	−0.250	11.832	
		5	10.920 62*	10.933 01	10.903 79	11.996 62	10.920 38
			0.002	0.116	−0.152	9.855	
2.10	−0.420	0	0.913 02	0.957 42	0.613 14	0.927 68*	0.929 03
			−1.722	3.056	−34.002	−0.144	
		1	2.873 99*	2.939 01	2.785 38	2.390 31	2.866 75
			0.253	2.521	−2.838	−16.619	
		2	4.859 00*	4.910 93	4.810 91	5.392 43	4.861 26
			−0.046	1.022	−1.036	10.927	
		3	6.850 51*	6.890 57	6.819 01	7.844 16	6.849 03
			0.022	0.606	−0.438	14.529	
		4	8.844 86*	8.876 41	8.822 14	9.961 69	8.845 47
			−0.007	0.350	−0.264	12.619	
		5	10.840 77*	10.866 29	10.823 37	11.971 65	10.840 28
			0.005	0.240	−0.156	10.437	
2.15	−0.645	0	0.867 79	0.955 76	0.566 30	0.882 91*	0.891 98
			−2.712	7.151	−36.512	−1.016	
		1	2.809 85*	2.915 50	2.719 12	2.278 24	2.798 61
			0.402	4.177	−2.840	−18.594	
		2	4.787 58*	4.869 38	4.738 07	5.389 00	4.791 18
			−0.075	1.632	−1.109	12.478	
		3	6.774 96*	6.837 36	6.742 46	7.837 34	6.772 64
			0.034	0.956	−0.446	15.721	
		4	8.766 57*	8.815 43	8.743 10	9.945 11	8.767 56
			−0.011	0.546	−0.279	13.431	
		5	10.760 49*	10.799 89	10.742 50	11.946 96	10.759 72
			0.007	0.373	−0.160	11.034	
2.20	−0.880	0	0.821 47	0.973 44	0.518 34	0.835 75*	0.853 90
			−3.798	13.999	−39.297	−2.125	
		1	2.745 01*	2.896 86	2.652 08	2.161 70	2.729 51
			0.568	6.131	−2.837	−20.803	
		2	4.715 58*	4.829 87	4.664 62	5.390 00	4.720 69
			−0.108	2.313	−1.188	14.178	
		3	6.698 91*	6.785 18	6.665 38	7.831 58	6.695 67
			0.048	1.337	−0.452	16.965	
		4	8.687 83*	8.755 04	8.663 59	9.928 97	8.689 25
			−0.016	0.757	−0.295	14.267	
		5	10.679 80*	10.733 83	10.661 21	11.922 53	10.678 71
			0.010	0.516	−0.164	11.648	

Table 5. WKB, SWKB and exact eigenenergies for the potential (13). The first line against each quantum number gives the energies and the second, the corresponding percentage errors, except for $n = 0$. The best values are marked by an asterisk.

a	b	n	1WKB	2WKB	1SWKB	2SWKB	Exact
0.60	0.10	0	−0.215 23	−0.282 55	0.000 00*		0.000 00
		1	2.975 33	2.959 30	3.122 74	3.074 44*	3.066 19
			−2.963	−3.486	1.844	0.269	
		2	7.121 08	7.112 82	7.242 06	7.196 57*	7.194 98
			−1.027	−1.142	0.654	0.022	
		3	11.978 11	11.972 80	12.083 54	12.041 87*	12.041 13
			−0.523	−0.567	0.352	0.006	
		4	17.428 06	17.424 26	17.522 87	17.484 43*	17.483 96
			−0.320	−0.341	0.223	0.003	
		5	23.397 2	23.394 3	23.484 1	23.448 3*	23.448 0
			−0.217	−0.229	0.154	0.001	
		6	29.833 8	29.831 5	29.914 6	29.881 0*	29.880 7
			−0.157	−0.165	0.113	0.001	
		7	36.699 2	36.697 3	36.775 0	36.743 2*	36.742 9
			−0.119	−0.124	0.087	0.001	
		8	43.962 8	43.961 2	44.034 4	44.004 2*	44.004 0
			−0.094	−0.097	0.069	0.000	
		9	51.599 8	51.598 4	51.667 9	51.639 1*	51.638 8
			−0.076	−0.078	0.056	0.001	
		10	59.589 6	59.588 4	59.654 7	59.627 0*	59.626 8
			−0.062	−0.064	0.047	0.000	
1.00	0.20	0	−0.269 49	−0.357 43	0.000 00*		0.000 00
		1	4.729 85	4.704 10	4.925 34	4.868 56*	4.861 10
			−2.700	−3.230	1.322	0.153	
		2	11.028 21	11.014 23	11.191 49	11.134 49*	11.132 63
			−0.938	−1.064	0.529	0.017	
		3	18.310 73	18.301 52	18.454 36	18.400 58*	18.399 69
			−0.483	−0.534	0.297	0.005	
		4	26.419 0	26.412 3	26.549 0	26.498 5*	26.497 9
			−0.298	−0.323	0.193	0.002	
		5	35.252 9	35.247 7	35.372 5	35.325 0*	35.324 6
			−0.203	−0.218	0.136	0.001	
		6	44.741 6	44.737 4	44.853 1	44.808 1*	44.807 8
			−0.148	−0.157	0.101	0.001	
		7	54.831 7	54.828 3	54.936 6	54.893 8*	54.893 5
			−0.113	−0.119	0.079	0.001	
		8	65.481 1	65.478 2	65.580 5	65.539 5*	65.539 2
			−0.089	−0.093	0.063	0.000	
		9	76.655 4	76.652 8	76.750 0	76.710 7*	76.710 5
			−0.072	−0.075	0.051	0.000	
		10	88.325 7	88.323 5	88.416 2	88.378 5*	88.378 2
			−0.059	−0.062	0.043	0.000	
1.25	0.50	0	−0.504 97	−0.659 27	0.000 00*		0.000 00
		1	6.344 16	6.310 53	6.681 74	6.565 93*	6.543 97
			−3.053	−3.567	2.105	0.336	
		2	15.358 16	15.341 17	15.633 31	15.526 98*	15.523 19
			−1.063	−1.173	0.709	0.024	
		3	25.974 5	25.963 7	26.213 5	26.117 2*	26.115 4
			−0.540	−0.581	0.376	0.007	
		4	37.923 7	37.916 0	38.138 1	38.049 8*	38.048 7
			−0.329	−0.349	0.235	0.003	
		5	51.038 4	51.032 6	51.234 7	51.152 8*	51.152 0
			−0.222	−0.233	0.162	0.002	

Table 5. (continued)

a	b	n	1WKB	2WKB	1SWKB	2SWKB	Exact
		6	65.202 0	65.197 3	65.384 2	65.307 5*	65.306 8
			−0.160	−0.168	0.119	0.001	
		7	80.326 9	80.323 1	80.497 7	80.425 4*	80.424 7
			−0.122	−0.126	0.091	0.001	
		8	96.344 4	96.341 2	96.505 7	96.437 1*	96.436 5
			−0.096	−0.099	0.072	0.001	
		9	113.199	113.196	113.352	113.287*	113.286
			−0.077	−0.079	0.058	0.001	
		10	130.843	130.841	130.990	130.927*	130.926
			−0.063	−0.065	0.049	0.001	
2.00	1.00	0	−0.655 52	−0.863 80	0.000 00*		0.000 00
		1	9.758 84	9.705 86	10.216 13	10.071 66*	10.048 89
			−2.886	−3.414	1.664	0.227	
		2	23.159 9	23.132 1	23.537 2	23.398 4*	23.393 6
			−0.999	−1.118	0.614	0.021	
		3	38.796 0	38.778 1	39.125 8	38.997 5*	38.995 3
			−0.511	−0.557	0.335	0.006	
		4	56.299 0	56.286 1	56.596 1	56.477 1*	56.475 7
			−0.313	−0.336	0.213	0.002	
		5	75.438 2	75.428 3	75.710 9	75.599 7*	75.598 7
			−0.212	−0.225	0.148	0.001	
		6	96.051 7	96.043 7	96.305 3	96.200 7*	96.199 8
			−0.154	−0.162	0.110	0.001	
		7	118.018	118.011	118.256	118.157*	118.156
			−0.117	−0.123	0.085	0.001	
		8	141.241	141.235	141.466	141.372*	141.371
			−0.092	−0.096	0.067	0.001	
		9	165.642	165.638	165.857	165.766*	165.766
			−0.075	−0.077	0.055	0.000	
		10	191.158	191.154	191.363	191.276*	191.275
			−0.061	−0.063	0.046	0.001	
3.00	2.00	0	−0.887 40	−1.173 69	0.000 00*		0.000 00
		1	14.392 31	14.313 88	15.024 28	14.833 06*	14.805 64
			−2.792	−3.321	1.477	0.185	
		2	33.837 1	33.795 2	34.361 8	34.173 9*	34.167 6
			−0.967	−1.090	0.568	0.018	
		3	56.418 2	56.390 9	56.878 4	56.702 9*	56.699 9
			−0.497	−0.545	0.315	0.005	
		4	81.625 0	81.605 3	82.040 4	81.876 6*	81.874 8
			−0.305	−0.329	0.202	0.002	
		5	109.136	109.121	109.518	109.364*	109.363
			−0.208	−0.221	0.142	0.001	
		6	138.725	138.713	139.080	138.935*	138.934
			−0.150	−0.159	0.105	0.001	
		7	170.221	170.211	170.555	170.417*	170.416
			−0.114	−0.120	0.082	0.001	
		8	203.490	203.482	203.807	203.675*	203.674
			−0.090	−0.094	0.065	0.000	
		9	238.423	238.415	238.724	238.598*	238.597
			−0.073	−0.076	0.053	0.000	
		10	274.927	274.921	275.215	275.094*	275.093
			−0.060	−0.063	0.044	0.000	

Table 6. WKB, SWKB and exact eigenenergies for the potential (16). The first line against each quantum number gives the energies and the second, the corresponding percentage errors, except for $n=0$. The best values are marked by an asterisk.

a	b	n	1WKB	2WKB	1SWKB	2SWKB	Exact
0.20	1.00	0	0.402 84	−0.018 09	0.000 00*		0.000 00
		1	2.521 55	2.575 38	2.483 25	2.601 23*	2.641 85
			−4.554	−2.516	−6.003	−1.538	
		2	3.311 31*	2.756 96	3.294 17	2.954 06	3.318 81
			−0.226	−16.929	−0.742	−10.990	
		3	4.096 84*	2.944 01	4.085 06	4.589 95	4.116 21
			−0.471	−28.478	−0.757	11.509	
		4	4.886 23*	3.251 89	4.877 33	5.525 31	4.889 24
			−0.062	−33.489	−0.244	13.010	
		5	5.678 26*	4.354 24	5.671 20	6.335 62	5.684 93
			−0.117	−23.407	−0.242	11.446	
		6	6.472 06*	5.642 12	6.466 26	7.112 21	6.473 78
			−0.027	−12.847	−0.116	9.862	
		7	7.267 08*	6.621 84	7.262 20	7.877 92	7.270 17
			−0.043	−8.918	− 0.110	8.360	
		8	8.062 97*	7.516 57	8.058 79	8.640 99	8.064 08
			−0.014	−6.789	−0.066	7.154	
		9	8.859 52*	8.375 76	8.855 88	9.404 75	8.861 21
			−0.019	−5.478	−0.060	6.134	
		10	9.656 56*	9.216 93	9.653 36	10.170 56	9.657 32
			−0.008	−4.560	−0.041	5.315	
0.50	1.00	0	0.340 27	−0.000 28	0.000 00*		0.000 00
		1	4.421 75	4.422 94	4.328 07	4.448 29*	4.447 32
			−0.575	−0.548	−2.681	0.022	
		2	6.957 36*	6.714 31	6.926 55	6.954 28	6.998 40
			−0.586	−4.059	−1.027	−0.630	
		3	9.168 34*	8.650 64	9.152 43	9.167 67	9.178 49
			−0.111	−5.751	−0.284	−0.118	
		4	11.286 61*	10.661 38	11.276 44	11.354 30	11.294 90
			−0.073	−5.609	−0.163	0.526	
		5	13.364 89*	12.750 35	13.357 64	13.498 05	13.368 48
			−0.027	−4.624	−0.081	0.969	
		6	15.421 71*	14.863 80	15.416 18	15.600 20	15.424 76
			− 0.020	−3.637	−0.056	1.137	
		7	17.465 41*	16.972 25	17.461 01	17.671 64	17.467 16
			−0.010	−2.833	−0.035	1.171	
		8	19.500 41*	19.066 55	19.496 79	19.721 96	19.501 90
			−0.008	−2.232	−0.026	1.128	
		9	21.529 3*	21.146 0	21.526 2	21.757 9	21.530 3
			−0.005	−1.785	−0.019	1.057	
		10	23.553 6*	23.212 5	23.551 0	23.784 0	23.554 4
			−0.003	−1.452	−0.014	0.975	
1.00	1.00	0	0.273 38	0.010 66	0.000 00*		0.000 00
		1	6.876 33	6.844 49	6.765 97	6.847 08*	6.851 40
			0.364	−0.101	−1.247	−0.063	
		2	12.079 95	12.034 63	12.031 83	12.091 37*	12.102 41
			−0.186	−0.560	−0.583	−0.091	
		3	16.728 38	16.590 83	16.702 98	16.730 27*	16.742 99
			−0.087	−0.909	−0.239	−0.076	
		4	21.136 7	20.921 3	21.121 1	21.137 5*	21.146 4
			−0.046	−1.064	−0.120	−0.042	
		5	25.422 0	25.162 0	25.411 5	25.433 1*	25.428 3
			−0.025	−1.047	−0.066	0.019	

Table 6. (continued)

a	b	n	1WKB	2WKB	1SWKB	2SWKB	Exact
		6	29.635 6*	29.357 4	29.628 0	29.661 9	29.640 0
			−0.015	−0.953	−0.040	0.074	
		7	33.803 2*	33.523 6	33.797 5	33.845 2	33.806 3
			−0.009	−0.836	−0.026	0.115	
		8	37.939 3*	37.667 7	37.934 7	37.995 5	37.941 6
			−0.006	−0.722	−0.018	0.142	
		9	42.052 6*	41.793 9	42.048 9	42.120 7	42.054 4
			−0.004	−0.619	−0.013	0.158	
		10	46.149 0*	45.905 1	46.145 9	46.226 6	46.150 4
			−0.003	−0.532	−0.010	0.165	
2.00	1.00	0	0.198 45	0.009 20	0.000 00*		0.000 00
		1	11.262 17	11.206 88*	11.155 99	11.204 68	11.207 26
			0.490	−0.003	−0.457	−0.023	
		2	21.197 4	21.180 5	21.137 3	21.187 5*	21.190 6
			0.032	−0.048	−0.252	−0.015	
		3	30.501 5	30.480 4	30.464 6	30.502 4*	30.506 3
			−0.016	−0.085	−0.137	−0.013	
		4	39.438 8	39.398 9	39.414 4	39.439 8*	39.445 4
			−0.017	−0.118	−0.079	−0.014	
		5	48.148 5*	48.087 6	48.131 5	48.148 1	48.154 6
			−0.013	−0.139	−0.048	−0.013	
		6	56.708 0*	56.628 8	56.695 5	56.707 3	56.713 1
			−0.009	−0.149	−0.031	−0.010	
		7	65.163 1*	65.069 8	65.153 6	65.163 6	65.167 3
			−0.006	−0.150	−0.021	−0.006	
		8	73.542 4	73.438 9	73.534 9	73.545 2*	73.545 8
			−0.005	−0.145	−0.015	−0.001	
		9	81.864 7*	81.754 6	81.858 7	81.870 6	81.867 5
			−0.003	−0.138	−0.011	0.004	
		10	90.142 9*	90.029 0	90.138 0	90.152 4	90.145 2
			−0.003	−0.129	−0.008	0.008	

4.2. Second potential, equation (13)

The results for the second, third and fourth potentials are shown in tables 5, 6, and 7 respectively. The tabular arrangement of the results is similar to that of table 3. For these three potentials, the ground state is zero and 1SWKB gives the exact result. As the ground-state energy is zero no percentage errors are shown for this level. For $n>0$ the percentage errors are shown immediately below the WKB and SWKB results. The best value amongst the four WKB and SWKB results is marked by an asterisk.

For the second potential we notice from table 5 that the inclusion of the second term has led to a worsening in the energy value for WKB, but an improvement for the SWKB. The 1SWKB results are better than 1WKB results, and 2SWKB results are better than 2WKB results. Here also we find that the convergence of the SWKB series is better than that of the WKB series. The 2SWKB results are the best in all cases.

4.3. Third potential, equation (16)

It can be shown that the Hamiltonian for this potential has a scaling property. If the eigenvalue is known for a certain value of the ratio b/a, eigenvalues for all other sets

Table 7. WKB, SWKB and exact eigenenergies for the potential (19). The first line against each quantum number gives the energies and the second, the corresponding percentage errors, except for $n = 0$. The best values are marked by an asterisk.

a	b	n	1WKB	2WKB	1SWKB	2SWKB	Exact
0.20	0.25	0	0.002 08	−0.000 01	0.000 00*		0.000 00
		1	0.893 65	0.891 57	0.891 59	0.891 59*	0.891 59
			0.231	−0.002	0.000	0.000	
		2	1.768 19	1.766 33*	1.766 24	1.766 35*	1.766 34
			0.105	−0.001	−0.006	0.001	
		3	2.626 07	2.624 61*	2.624 29	2.624 62*	2.624 61
			0.056	0.000	−0.012	0.000	
		4	3.468 35	3.467 40*	3.466 80	3.467 40*	3.467 39
			0.028	0.000	−0.017	0.000	
		5	4.296 75	4.296 33*	4.295 47	4.296 33*	4.296 32
			0.010	0.000	−0.020	0.000	
		6	5.113 45	5.113 49*	5.112 44	5.113 49*	5.113 50
			−0.001	0.000	−0.021	0.000	
		7	5.920 83	5.921 18	5.920 06	5.921 20*	5.921 23
			−0.007	−0.001	−0.020	−0.001	
		8	6.721 17	6.721 71	6.720 62	6.721 74*	6.721 78
			−0.009	−0.001	−0.017	−0.001	
		9	7.516 50	7.517 13	7.516 13	7.517 17*	7.517 19
			−0.009	−0.001	−0.014	0.000	
		10	8.308 48	8.309 14*	8.308 25	8.309 18*	8.309 16
			−0.008	0.000	−0.011	0.000	
0.20	0.30	0	0.007 52	−0.000 16	0.000 00*		0.000 00
		1	0.976 23	0.968 86	0.969 03	0.969 03*	0.969 04
			0.742	−0.019	−0.001	−0.001	
		2	1.880 71	1.876 50*	1.874 85	1.876 52	1.876 38
			0.231	0.006	−0.082	0.007	
		3	2.727 37*	2.727 60	2.723 49	2.727 54	2.727 36
			0.000	0.009	−0.142	0.007	
		4	3.533 28	3.535 77	3.531 23	3.535 98*	3.536 23
			−0.083	−0.013	−0.141	−0.007	
		5	4.317 71	4.320 51	4.316 88	4.320 80*	4.321 03
			−0.077	−0.012	−0.096	−0.005	
		6	5.094 11	5.097 34	5.093 92	5.097 06*	5.096 47
			−0.046	0.017	−0.050	0.012	
		7	5.869 39	5.873 56	5.869 51*	5.872 30	5.870 76
			−0.023	0.048	−0.021	0.026	
		8	6.646 51	6.651 61	6.646 73*	6.649 29	6.647 19
			−0.010	0.066	−0.007	0.032	
		9	7.426 40	7.432 01	7.426 64*	7.428 79	7.426 67
			−0.004	0.072	0.000	0.029	
		10	8.209 14*	8.214 72	8.209 37	8.210 88	8.209 19
			−0.001	0.067	0.002	0.021	
0.20	0.40	0	0.025 18	−0.001 25	0.000 00*		0.000 00
		1	1.111 02	1.088 98	1.089 73	1.090 68*	1.090 59
			1.873	−0.148	−0.079	0.008	
		2	1.959 91	1.972 42*	1.952 94	1.974 10	1.971 94
			−0.610	0.024	−0.964	0.110	
		3	2.707 32	2.723 74	2.707 02	2.721 64*	2.722 31
			−0.551	0.053	−0.562	−0.025	
		4	3.455 31	3.490 70	3.456 04*	3.470 18	3.456 80
			−0.043	0.981	−0.022	0.387	
		5	4.216 91*	4.251 63	4.217 55	4.225 13	4.216 16
			0.018	0.841	0.033	0.213	

Table 7. (continued)

a	*b*	*n*	1WKB	2WKB	1SWKB	2SWKB	Exact
		6	4.989 49*	5.011 90	4.989 89	4.988 03	4.988 97
			0.010	0.460	0.018	−0.019	
		7	5.769 58*	5.776 59	5.769 80	5.759 41	5.768 92
			0.011	0.133	0.015	−0.165	
		8	6.554 66*	6.547 37	6.554 77	6.538 38	6.554 20
			0.007	−0.104	0.009	−0.241	
		9	7.343 09*	7.324 21	7.343 13	7.323 54	7.342 78
			0.004	−0.253	0.005	−0.262	
		10	8.133 82*	8.106 44	8.133 82*	8.113 46	8.133 60
			0.003	−0.334	0.003	−0.248	
0.20	0.50	0	0.048 77	−0.003 21	0.000 00*		0.000 00
		1	1.198 55	1.197 53	1.168 14*	1.185 62	1.175 55
			1.957	1.870	−0.630	0.857	
		2	1.919 25	1.991 92	1.919 35	1.962 27*	1.964 71
			−2.314	1.385	−2.309	−0.124	
		3	2.642 59*	2.770 40	2.643 67	2.676 69	2.639 41
			0.120	4.963	0.161	1.412	
		4	3.397 17	3.459 60	3.397 59	3.389 76*	3.392 19
			0.147	1.987	0.159	−0.072	
		5	4.168 90	4.164 44	4.168 92*	4.132 18	4.169 86
			−0.023	−0.130	−0.023	−0.904	
		6	4.949 89	4.896 37	4.949 73*	4.900 54	4.948 66
			0.025	−1.057	0.022	−0.972	
		7	5.736 14	5.652 70	5.735 93*	5.687 06	5.735 95
			0.003	−1.451	0.000	−0.852	
		8	6.525 63*	6.427 57	6.525 41	6.484 50	6.525 53
			0.002	−1.501	−0.002	−0.629	
		9	7.317 24	7.215 11	7.317 03*	7.287 46	7.317 08
			0.002	−1.394	−0.001	−0.405	
		10	8.110 35*	8.010 66	8.110 15	8.092 51	8.110 33
			0.000	−1.229	−0.002	−0.220	
0.20	0.60	0	0.076 15	−0.005 09	0.000 00*		0.000 00
		1	1.213 83*	1.212 51	1.198 42	1.284 60	1.230 77
			−1.376	−1.484	−2.628	4.374	
		2	1.872 49	2.189 35	1.873 79*	1.972 89	1.916 48
			−2.295	14.238	−2.228	2.943	
		3	2.605 38	2.719 77	2.605 36	2.581 38*	2.578 50
			1.042	5.479	1.042	0.112	
		4	3.369 35*	3.316 78	3.368 76	3.279 85	3.374 34
			−0.148	−1.706	−0.165	−2.800	
		5	4.147 11*	3.990 80	4.146 43	4.046 59	4.147 43
			−0.008	−3.777	−0.024	−2.431	
		6	4.931 81	4.728 31	4.931 17*	4.851 15	4.930 25
			0.032	−0.4096	0.019	−1.604	
		7	5.720 46*	5.508 20	5.719 90	5.670 34	5.721 51
			−0.018	−3.728	−0.028	−0.894	
		8	6.511 61	6.311 59	6.511 11*	6.491 22	6.511 09
			0.008	−3.064	0.000	−0.305	
		9	7.304 45*	7.125 75	7.304 02	7.308 64	7.304 66
			−0.003	−2.449	−0.009	0.054	
		10	8.098 51*	7.943 60	8.098 13	8.121 45	8.098 49
			0.000	−1.913	−0.004	0.284	

of the parameters a and b which have the same value for the ratio b/a can be obtained by multiplying the known eigenvalue by a function of a and b. Because of the scaling property, the parameter b was kept fixed at 1 and a was given values from 0.2 to 1.0. For this potential we find a complicated pattern in the results. For the first two sets we notice from table 6 that 2WKB results become worse than the 1WKB ones after $n=1$, and a similar statement holds for the SWKB results. For $n<2$, the 2SWKB results are the best, but above this the 1WKB results are the best. But at $a=1.0$, $b=1.0$, the pattern changes. 2WKB is worse than 1WKB for $n>1$, but the dividing line for the SWKB case is now at $n=5$. Also the pattern of 'best results' is different from the previous two cases. For $n<6$, 2SWKB results are the best, but above it, 1WKB ones are the best. The pattern of results for $a=2$, $b=1$ is similar to that of the previous set, except for two minor differences.

4.4. Fourth potential, equation (19)

The Hamiltonian for this potential also has a scaling property similar to that of the potential (16). Hence the parameter a was kept fixed at 0.2 and b was given values between 0.25 and 0.6. We notice from table 7 that in the first set ($b=0.25$), the 2WKB results are better than the 1WKB ones, and a similar situation holds for the SWKB results. But in the next set, we notice that for $n<7$, 2WKB is better than 1WKB, and 2SWKB is better than 1SWKB, but from $n=7$ onward the reverse pattern is observed. This dividing line shifts to smaller n values as b is further increased. At $b=0.6$ it has reached $n=1$. The pattern of 'best results' is quite complicated and is best seen by referring to the asterisks in table 7.

5. Conclusion

The results presented in this paper show that while in certain situations the effect of the second term in the WKB and SWKB approximations may be uniform and the convergence of the SWKB series may be better than the WKB series, these are by no means universal results. In general, the effect of the second term in the WKB and SWKB approximations depends on all the factors involved, namely the potential, the parameters involved and the quantum number. No simple generalizations are possible. In some cases 1SWKB can be better than 1WKB, and 2SWKB can be better than 2WKB but the reverse can be true in some other cases.

Acknowledgment

This work was supported in part by a research grant from the Natural Sciences and Engineering Research Council of Canada to the author.

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